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LEVEL II

A GENERAL MOMENT INEQUALITY FOR THE MAXIMUM OF RECTANGULAR PARTIAL SUMS OF MULTIPLE SERIES

by

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A GENERAL MOMENT INEQUALITY FOR THE MAXIMUM OF RECTANGULAR PARTIAL SUMS OF MULTIPLE SERIES

ABSTRACT

In a recent paper [7] we presented a general method of how to obtain an upper estimate *for a fixed moment* of the maximum of partial sums of a single series in terms of the given "a priori" upper estimate for the same moment of the partial sums. Now we extend this method from single series to multiple series.

Let Z_+^d be the set of all d-tuples $k = (k_1,\ldots,k_d)$ with nonnegative integers for coordinates; if all k_j are positive, we write $k \in Z_1^d$. Denote by $R = R(b,m) = R(b_1,\ldots,b_d; m_1,\ldots,m_d)$ the rectangle $\mathop{X}_{j=1}^{d} (b_j, b_j+m_j]$ in Z_1^d, where $b \in Z_+^d$ and $m \in Z_1^d$. Considering a d-multiple sequence of functions $\{\xi_k = \xi_k(x) : k \in Z_1^d\} \subset L^\gamma(X,A,\mu)$, where $\gamma \geq 1$ is a fixed real, set

$$S(b,m) = S(R) = \sum_{k\in R} \xi_k = \sum_{k_1=b_1+1}^{b_1+m_1} \cdots \sum_{k_d=b_d+1}^{b_d+m_d} \xi_{k_1,\ldots,k_d}$$

and

$$M(b,m) = M(R) = \max_{1\leq p_1\leq m_1} \cdots \max_{1\leq p_d\leq m_d} |S(b_1,\ldots,b_d; p_1,\ldots,p_d)|.$$

Our main result is that, under very mild assumptions on the nonnegative functions $f(R) = f(b_1,\ldots,b_d; m_1,\ldots,m_d)$ and $\phi(t; m_1,\ldots,m_d)$, $t \geq 0$ real, if we have for every rectangle R in Z_1^d the inequality

$$\int |S(R)|^\gamma d\mu \leq f(R)\, \phi^\gamma(f(R); m_1,\ldots,m_d),$$

then we have also for every rectangle R the inequality

$$\int M^\gamma(R) d\mu \leq 3^{d(\gamma-1)} f(R) \times$$

$$\times \left\{ \sum_{k_1=0}^{[\log m_1]} \cdots \sum_{k_d=0}^{[\log m_d]} \phi\left(\frac{f(R)}{2^{k_1+\cdots+k_d}} ; \left[\frac{m_1}{2^{k_1}}\right], \ldots, \left[\frac{m_d}{2^{k_d}}\right]\right)\right\}^\gamma .$$

The integrals are taken over X, $[\cdot]$ denotes the integral part, and the logarithms are with base 2.

A number of special cases interesting in themselves are included.

A General Moment Inequality for the Maximum of Rectangular Partial Sums of Multiple Series

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1. A Preliminary Result

Let (X,A,μ) be a positive measure space and let $\{\xi_{k_1} = \xi_{k_1}(x) : k_1 \in Z_1\} \subset L^\gamma(X,A,\mu)$ where $Z_1 = \{1,2,\ldots\}$ and γ is a fixed real, $\gamma \geq 1$. Studying the a.e. convergence of the single series

$$\sum_{k_1=1}^{\infty} \xi_{k_1}, \tag{1.1}$$

denote by $S(I)$ and $M(I)$ the partial sum of (1.1) extended over the integers contained in the interval $I = (b_1, b_1 + m_1]$ and the maximum of the consecutive partial sums extended also over I, respectively. That is,

$$S(I) = S(b_1,m_1) = \sum_{k_1\in I} \xi_{k_1} = \sum_{k_1=b_1+1}^{b_1+m_1} \xi_{k_1}$$

and

$$M(I) = M(b_1,m_1) = \max_{1\leq p_1\leq m_1} |S(b_1,p_1)|.$$

Here and in the sequel $b_1 \in Z_+ = \{0,1,\ldots\}$ and $p_1, m_1 \in Z_1$; further, $m_1 = |I|$ denotes the number of the integers contained in the interval I. We note that clearly

$$M(I) \leq \max_{J\subseteq I} |S(J)| \leq 2M(I).$$

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A nonnegative function $f(I)$ of the interval I with integral endpoints is said to be superadditive if for every I and for every disjoint representation

$$I_1 \cup I_2 = I$$

we have the inequality

$$f(I_1) + f(I_2) \le f(I).$$

Further, let $\phi(t_1, m_1)$ be also a nonnegative function defined on $R_+ \times Z_1$ where R_+ is the set of the nonnegative reals.

A recent result by the present author (1980) reads as follows.

THEOREM 1 ([7]). Let $\gamma \ge 1$ be given. Suppose that there exist a nonnegative and superadditive function $f(I)$ of the interval I, and a nonnegative function $\phi(t_1, m_1)$, nondecreasing in both variables, such that for every I we have

$$\int |S(I)|^\gamma d\mu \le f(I)\, \phi^\gamma(f(I), m_1), \quad m_1 = |I|.$$

Then for every I we have both

$$\int M^\gamma(I)\, d\mu \le 3^{\gamma-1} f(I) \left\{ \sum_{k_1=0}^{[\log m_1]-1} \phi\left(\frac{f(I)}{2^{k_1}}, \left[\frac{m_1}{2^{k_1+1}} \right] \right) \right\}^\gamma \tag{1.2}$$

and

$$\int M^\gamma(I)\, d\mu \le \frac{5}{2} f(I) \left\{ \sum_{k_1=0}^{[\log m_1]} \phi\left(\frac{f(I)}{2^{k_1}}, \left[\frac{m_1}{2^{k_1}} \right] \right) \right\}^\gamma .$$

In this paper the integrals are taken over the whole space X, $[t_1]$ is the integral part of t_1, and the logarithms are with base 2. Furthermore, in the case $m_1 = 1$ we agree to take $[\log m_1] - 1$ to be equal to 0 and $[m_1/2^{k_1+1}]$ to be equal to 1 on the righthand side of (1.2).

2. The Main Result

Let Z_+^d be the set of all d-tuples $k = (k_1, \ldots, k_d)$ with nonnegative integers for coordinates, where the dimension d is a fixed positive integer. As usual, $k \le m$ iff $k_j \le m_j$ for each j, and we write $1 = (1, \ldots, 1)$. If all the coordinates k_j are positive integers, we write $k \in Z_1^d$.

Let $\{\xi_k = \xi_k(x) : k \in Z_1^d\} \subset L^\gamma(X, A, \mu)$ be given and consider the d-multiple series

$$\sum_{k \in Z_1^d} \xi_k = \sum_{k_1=1}^{\infty} \cdots \sum_{k_d=1}^{\infty} \xi_{k_1, \ldots, k_d}. \tag{2.1}$$

In the following, we denote by

$$R = R(b,m) = R(b_1, \ldots, b_d;\, m_1, \ldots, m_d) =$$

$$= \{k \in Z_1^d : b_j < k_j \le b_j + m_j \text{ for each } j,\ 1 \le j \le d\} = \mathop{\times}_{j=1}^{d} (b_j, b_j + m_j]$$

an arbitrary rectangle in Z_1^d where $b \in Z_+^d$ and $m \in Z_1^d$. The rectangular partial sum $S(R)$ of (2.1) extended over the lattice points contained in R, and the maximum $M(R)$ extended over R to those rectangular partial sums whose lefthand bottom corners coincide with that of R, are defined as follows:

$$S(R) = S(b,m) = S(b_1, \ldots, b_d;\, m_1, \ldots, m_d) =$$

$$= \sum_{k \in R} \xi_k = \sum_{k_1=b_1+1}^{b_1+m_1} \cdots \sum_{k_d=b_d+1}^{b_d+m_d} \xi_{k_1, \ldots, k_d}$$

and

$$M(R) = M(b,m) = M(b_1, \ldots, b_d;\, m_1, \ldots, m_d) =$$

$$= \max_{1 \le p \le m} |S(b,p)| = \max_{1 \le p_1 \le m_1} \cdots \max_{1 \le p_d \le m_d} |S(b_1, \ldots, b_d;\, p_1, \ldots, p_d)|,$$

respectively. Here and in the sequel $b \in Z_+^d$ and $m \in Z_1^d$; further, m_j

denotes the number of the lattice points contained in the rectangle R in a row parallel to the j^{th} axis, $1 \le j \le d$. We note that clearly

$$M(R) \le \max_{Q \subseteq R} |S(Q)| \le 2^d M(R).$$

A nonnegative function $f(R)$ of the rectangle R with corner points from Z_+^d is said to be superadditive if we have the inequality

$$f(R_{j1}) + f(R_{j2}) \le f(R) \tag{2.2}$$

for every rectangle R and for every j and p_j where $1 \le j \le d$, $1 \le p_j < m_j$, and

$$R_{j1} = R(b_1,\dots,b_{j-1},b_j,b_{j+1},\dots,b_d;\ m_1,\dots,m_{j-1},p_j,m_{j+1},\dots,m_d),$$

$$R_{j2} = R(b_1,\dots,b_{j-1},b_j+p_j,b_{j+1},\dots,b_d;\ m_1,\dots,m_{j-1},m_j-p_j,m_{j+1},\dots,m_d).$$

In other words,

$$R_{j1} \cup R_{j2} = R$$

is a disjoint decomposition of R by a hyperplane parallel to each axis except the j^{th} axis. For example,

$$f(R) = \sum_{k \in R} u_k$$

is even an additive function of R, where $\{u_k :\ k \in Z_1^d\}$ is a given d-multiple sequence of nonnegative reals. We mention that the nonnegativity of $f(R)$ and (2.2) imply that $f(R) = f(b_1,\dots,b_d;\ m_1,\dots,m_d)$ is a nondecreasing function in each variable m_j, $1 \le j \le d$.

Furthermore, by $\phi(t_1,m) = \phi(t_1;\ m_1,\dots,m_d)$ we denote a nonnegative function defined on $R_+ \times Z_1^d$, which is nondecreasing in each variable, i.e.

$$\phi(t_1';\ m_1',\dots,m_d') \le \phi(t_1'';\ m_1'',\dots,m_d'')$$

whenever

$$0 \le t_1' \le t_1'' \text{ and } 1 \le m_j' \le m_j'' \text{ for each } j,\ 1 \le j \le d.$$

After these preliminaries we give an upper estimate for the γ^{th} moment of $M(R)$ in the terms of the given "a priori" upper estimate for the γ^{th} moment of $S(R)$, while R runs over all the rectangles in Z_1^d. This generalization of Theorem 1 reads as follows.

THEOREM 2. <u>Let $\gamma \ge 1$ and $d \ge 1$ be given. Suppose that there exist a nonnegative and superadditive function of $f(R)$ of the rectangle R in Z_1^d, and a nonnegative function $\phi(t_1;\ m_1,\dots,m_d)$, nondecreasing in each variable, such that for every $R = R(b_1,\dots,b_d;\ m_1,\dots,m_d)$ we have</u>

$$\int |S(R)|^\gamma d\mu \le f(R)\, \phi^\gamma(f(R);\ m_1,\dots,m_d).$$

<u>Then for every R we have both the inequality</u>

$$\int M^\gamma(R)\, d\mu \le 3^{d(\gamma-1)} f(R) \times$$

$$\times \left\{ \sum_{k_1=0}^{[\log m_1]-1} \dots \sum_{k_d=0}^{[\log m_d]-1} \phi\left(\frac{f(R)}{2^{k_1+\dots+k_d}};\ \left[\frac{m_1}{2^{k_1+1}}\right],\dots,\left[\frac{m_d}{2^{k_d+1}}\right]\right)\right\}^\gamma \tag{2.3}$$

<u>and the inequality</u>

$$\int M^\gamma(R)\, d\mu \le \left(\frac{5}{2}\right)^d f(R) \times$$

$$\times \left\{ \sum_{k_1=0}^{[\log m_1]} \dots \sum_{k_d=0}^{[\log m_d]} \phi\left(\frac{f(R)}{2^{k_1+\dots+k_d}};\ \left[\frac{m_1}{2^{k_1}}\right],\dots,\left[\frac{m_d}{2^{k_d}}\right]\right)\right\}^\gamma. \tag{2.4}$$

Again we use the following convention: in case $m_j = 1$ for some j, $1 \le j \le d$, we take $[\log m_j] - 1$ to be equal to 0 and $[m_j/2^{k_j+1}]$ to be equal to 1 on the right of (2.3).

1. Special Cases

Without aiming at completeness we present here some special cases of Theorem 2 of interest in themselves.

Let us take $\phi(t_1; m_1,\dots,m_d) = t_1^{(\alpha-1)/\gamma}$ with real α, $\alpha > 1$. Then

$$\tilde{\Phi}_d\,(t_1; m_1,\dots,m_d) = \sum_{k_1=0}^{[\log m_1]} \cdots \sum_{k_d=0}^{[\log m_d]} \phi\left(\frac{t_1}{2^{k_1+\dots+k_d}};\left[\frac{m_1}{2^{k_1}}\right],\dots,\left[\frac{m_d}{2^{k_d}}\right]\right) \le$$

$$\le (1 - 2^{(1-\alpha)/\gamma})^{-d}\ \ t_1^{(\alpha-1)/\gamma},$$

independently of $m_1,\dots,m_d$.

COROLLARY 1. Let $\alpha > 1$, $\gamma \ge 1$, and $d \ge 1$ be given. Suppose that there exists a nonnegative and superadditive function $f(R)$ of the rectangle R in Z_1^d such that for every R we have

$$\int |S(R)|^\gamma\, d\mu \le f^\alpha(R).$$

Then for every R we have

$$\int M^\gamma(R)\, d\mu \le \left(\frac{5}{2}\right)^d (1 - 2^{(1-\alpha)/\gamma})^{-d\gamma} f^\alpha(R).$$

This result apart from the factor $(5/2)^d$ on the right was proved by the present author in [5, Theorem 7]. For $d=1$ see Longnecker and Serfling [3], and [4].

It is instructive to state this corollary for the *still more particular* case when $f(R) = \sum_{k\in R} u_k$, where $\{u_k;\ k \in Z_1^d\}$ is a d-multiple sequence of nonnegative reals.

COROLLARY 1a. (The d-multiple version of the Erdős-Stečkin inequality.) Let $\alpha > 1$, $\gamma \ge 1$, and $\{u_k \ge 0;\ k \in Z_1^d\}$ be given. Suppose that for every rectangle R in Z_1^d we have

$$\int |S(R)|^\gamma\, d\mu \le \Big(\sum_{k\in R} u_k\Big)^\alpha.$$

Then for every R we have

$$\int M^\gamma(R)\, d\mu \le \left(\frac{5}{2}\right)^d (1 - 2^{(1-\alpha)/\gamma})^{-d\gamma} \Big(\sum_{k\in R} u_k\Big)^\alpha.$$

As to the case $d=1$, see Erdős [1] and Gapoškin [2, pp. 29-31], the latter author making use of the oral communication of S. B. Stečkin.

Now take $\phi(t_1; m_1,\dots,m_d) = t_1^{(\alpha-1)/\gamma}\, w(t_1)$ where again $\alpha > 1$ and $w_1(t)$ is a slowly varying positive function, i.e. $w(t_1)$ is defined on R_+, $w(t_1) > 0$ for $t_1 > 0$, and for every positive C we have

$$\frac{w(Ct_1)}{w(t_1)} \to 1 \text{ as } t_1 \to \infty.$$

We emphasize that $w(t_1)$ is not necessarily a nondecreasing function, only $t_1^{(\alpha-1)/\gamma}\, w(t_1)$ has to be nondecreasing. For example,

$$w(t_1) = \{\log(1 + t_1)\}^\beta \{\log\log(2 + t_1)\}^\delta$$

is a slowly varying function, where β and δ are arbitrary reals. It is easy to check that again we have

$$\tilde{\Phi}_d(t_1; m_1,\dots,m_d) \le C(\alpha,\gamma,d,w)\ t_1^{(\alpha-1)/\gamma}\, w(t_1),$$

where $C(\alpha,\gamma,d,w)$ denotes a positive constant depending only on α, γ, d, and $w(t_1)$.

COROLLARY 2. Let $\alpha > 1$, $\gamma \ge 1$, and $d \ge 1$ be given. Suppose that there exist a nonnegative and superadditive function $f(R)$ of the rectangle R in Z_1^d, and a slowly varying positive function $w(t_1)$, $t_1^{(\alpha-1)/\gamma}\, w(t_1)$ is nondecreasing, such that for every R we have

$$\int |S(R)|^\gamma\, d\mu \le f^\alpha(R)\ w^\gamma(f(R)).$$

Then for every R we have

$$\int M^{\gamma}(R)\, d\mu \le \left(\frac{5}{2}\right)^{d} C^{\gamma}(\alpha,\gamma,d,w)\, f^{\alpha}(R)\, w^{\gamma}(f(R)).$$

Next take $\phi(t_1; m_1,\dots,m_d) = \lambda(m_1,\dots,m_d)$ where $\lambda(m_1,\dots,m_d)$ is defined on Z_1^d, positive and nondecreasing in each variable.

COROLLARY 3. Let $\gamma \ge 1$ and $d \ge 1$ be given. Suppose that there exist a nonnegative and superadditive function $f(R)$ of the rectangle R in Z_1^d, and a positive and nondecreasing d-multiple sequence $\{\lambda(m): m \in Z_1^d\}$ such that for every $R = R(b_1,\dots,b_d; m_1,\dots,m_d)$ we have

$$\int |S(R)|^{\gamma} d\mu \le f(R)\, \lambda^{\gamma}(m_1,\dots,m_d).$$

Then for every R we have

$$\int M^{\gamma}(R)\, d\mu \le 3^{d(\gamma-1)} f(R) \times$$

$$\times \left\{ \sum_{k_1=0}^{[\log m_1]-1} \cdots \sum_{k_d=0}^{[\log m_d]-1} \lambda\left(\left[\frac{m_1}{2^{k_1+1}}\right],\dots,\left[\frac{m_d}{2^{k_d+1}}\right]\right)\right\}^{\gamma}$$

with the same convention as in Theorem 2 in the case $m_j = 1$ for some j.

This moment inequality, apart from the factor $3^{d(\gamma-1)}$ on the right, was proved also by the present author in [6, Theorem 1] in a slightly different form.

To illuminate the strength of Corollary 3, we present two particular cases. First, assume that $\{\xi_k: k \in Z_1^d\}$ is a d-multiple orthogonal system. Then we obviously have

$$\int S^2(R)\, d\mu = \sum_{k\in R} \sigma_k^2 \quad \text{where} \quad \sigma_k^2 = \int \xi_k^2\, d\mu.$$

COROLLARY 3a. (The d-multiple version of the Rademacher-Menšov inequality.) If $\{\xi_k: k \in Z_1^d\}$ is a d-multiple orthogonal system, then for every rectangle $R = R(b_1,\dots,b_d; m_1,\dots,m_d)$ we have

$$\int M^2(R)\, d\mu \le 3^d \left(\sum_{k\in R} \sigma_k^2\right) \prod_{j=1}^{d} (\log(m_j+1))^2.$$

As to the case $d=1$, see e.g. [8, p.83].

Secondly, assume that $\phi(t_1; m_1,\dots,m_d) = \lambda(m_1,\dots,m_d)$ essentially grows in each variable in the sense that there exist on $m_0 \in Z_1$ and a real q, $q > 1$, such that for every j, $1 \le j \le d$, and for every $m \in Z_1^d$ with $m_j \ge m_0$ we have

$$\frac{\lambda(m_1,\dots,m_{j-1}, 2m_j, m_{j+1},\dots,m_d)}{\lambda(m_1,\dots,m_{j-1}, m_j, m_{j+1},\dots,m_d)} \ge q \qquad (3.1).$$

E.g. $\lambda(m) = \prod_{j=1}^{d} m_j^{\alpha_j} w_j(m_j)$ is such a d-multiple sequence where $\alpha_j > 0$ and $w_j(m_j)$ is a slowly varying function for each j, $1 \le j \le d$. Now (3.1) implies, in a routine way, that

$$\tilde{\Phi}_d(t_1; m_1,\dots,m_d) \le C(q,m_0)\, \lambda(m_1,\dots,m_d),$$

where the positive constant $C(q,m_0)$ depends only on q and on those values $\lambda(m)$ for which $m_j \le m_0$ for each j, $1 \le j \le d$. In particular, $C(q,m_0) = \{q/(q-1)\}^d$ if $m_0 = 1$.

COROLLARY 3b. Let $\gamma \ge 1$ and $d \ge 1$ be given. Suppose that there exist a nonnegative and superadditive function $f(R)$ of the rectangle R in Z_1^d and a d-multiple positive sequence $\{\lambda(m): m \in Z_1^d\}$ satisfying relation (3.1) with a $q > 1$ such that for every $R = R(b_1,\dots,b_d; m_1,\dots,m_d)$ we have

$$\int |S(R)|^{\gamma} d\mu \le f(R)\, \lambda^{\gamma}(m_1,\dots,m_d).$$

Then for every R we have

$$\int M^{\gamma}(R)\, d\mu \le \left(\frac{5}{2}\right)^{d} C^{\gamma}(q,m_0)\, f(R)\, \lambda^{\gamma}(m_1,\dots,m_d).$$

The special case $d=1$, $\gamma=2$, $f(R) = |R|^{\gamma-2} \sum_{k \in R} a_k^{\gamma}$, and $\lambda(m_1) = m_1^{(\gamma-2)/\gamma}$

with the assumption that $|\xi_{k_1}| \le B$ *a.e.* $(k_1=1,2,\ldots)$ is known as the Menšov-Paley inequality (cf. [9, p. 189]).

Finally, it is worth mentioning that in any case we can conclude the following

COROLLARY 4. Under the conditions of Theorem 2, for every rectangle $R = R(b_1,\ldots,b_d;\ m_1,\ldots,m_d)$ we have

$$\int M^{\gamma}(R)\, d\mu \le 3^{d(\gamma-1)} f(R)\, \phi^{\gamma}\left(f(R);\ \left[\frac{m_1}{2}\right],\ldots,\left[\frac{m_d}{2}\right]\right) \prod_{j=1}^{d} (\log(m_j+1))^{\gamma},$$

where again in the case $m_j = 1$ for some j we take $[m_j/2]$ equal to 1.

4. Proof of Theorem 2

The proof proceeds by induction on d. The case $d=1$ is stated in Theorem 1.

Assume now that Theorem 2 holds for $d-1$. We will show *that it holds* for d. Consequently, the induction hypo*thesis can* be applied to the following "partial" maximum:

$$M_{d-1}(R) = M_{d-1}(b,m) = M_{d-1}(b_1,\ldots,b_d;\ m_1,\ldots,m_{d-1},m_d) =$$

$$\max_{1 \le p_1 \le m_1} \cdots \max_{1 \le p_{d-1} \le m_{d-1}} |S(b_1,\ldots,b_d;\ p_1,\ldots,p_{d-1},m_d)|.$$

Proof of (2.3). By the *induction hypothesis,*

$$\int M_{d-1}^{\gamma}(R)\, d\mu \le 3^{(d-1)(\gamma-1)}\, f(R)\, \phi_{d-1}^{\gamma}(f(R);\ m_1,\ldots,m_d), \tag{4.1}$$

where

$$\phi_{d-1}(t_1;\ m_1,\ldots,m_{d-1},m_d) =$$

$$= \sum_{k_1=0}^{[\log m_1]-1} \cdots \sum_{k_{d-1}=0}^{[\log m_{d-1}]-1} \phi\left(\frac{t_1}{2^{k_1+\cdots+k_{d-1}}};\ \left[\frac{m_1}{2^{k_1+1}}\right],\ldots,\left[\frac{m_{d-1}}{2^{k_{d-1}+1}}\right],\ m_d\right)$$

and in the case $m_j = 1$ we mean 0 by $[\log m_j]-1$ and 1 by $[m_j/2^{k_j+1}]$.

Setting

$$\phi_d(t_1;\ m_1,\ldots,m_d) =$$

$$= \sum_{k_1=0}^{[\log m_1]-1} \cdots \sum_{k_d=0}^{[\log m_d]-1} \phi\left(\frac{t_1}{2^{k_1+\cdots+k_d}};\ \left[\frac{m_1}{2^{k_1+1}}\right],\ldots,\left[\frac{m_d}{2^{k_d+1}}\right]\right)$$

with the above convention, *inequality* (2.3) to be proved can be rewritten as follows:

$$\int M_d(R)\, d\mu \le 3^{d(\gamma-1)}\, f(R)\, \phi_d^{\gamma}(f(R);\ m_1,\ldots,m_d), \tag{4.2}$$

where

$$M_d(R) = M(R) = \max_{1 \le p_1 \le m_1} \cdots \max_{1 \le p_d \le m_d} |S(b_1,\ldots,b_d;\ p_1,\ldots,p_d)| =$$

$$= \max_{1 \le p_d \le m_d} M_{d-1}(R).$$

It is not hard to verify that $\phi_d(t_1;\ m_1,\ldots,m_d)$ can be also expressed by the aid of $\phi_{d-1}(t_1;\ m_1,\ldots,m_d)$ as follows:

$$\phi_d(t_1;\ m_1,\ldots,m_{d-1},m_d) =$$

$$= \sum_{k_d=0}^{[\log m_d]-1} \phi_{d-1}\left(\frac{t_1}{2^{k_d}};\ m_1,\ldots,m_{d-1},\left[\frac{m_d}{2^{k_d+1}}\right]\right),$$

with the same *convention as* above concerning the case $m_d=1$. *This relation* also turns into the following recurrence:

$$\phi_d(t_1;\ m_1,\ldots,m_d) = \phi_{d-1}(t_1;\ m_1,\ldots,m_d) \quad \text{for} \quad m_d = 1,2,3 \tag{4.3}$$

and

$$\phi_d(t_1;\ m_1,\ldots,m_{d-1},m_d) = \phi_{d-1}\left(t_1;\ m_1,\ldots,m_{d-1},\left[\frac{m_d}{2}\right]\right) +$$

$$+\ \Phi_d\left(\frac{t_1}{2};\ m_1,\dots,m_{d-1},\left[\frac{m_d}{2}\right]\right) \quad \text{for } m_d \ge 4. \tag{4.4}$$

After these preliminaries we can prove (4.2) by using again an induction but this time on m_d. Both the case of the initial values $m_d = 1,2,3$ and the induction step are similar to the argument explained in the proof of Theorem 1 in [7]. Therefore, we only sketch the proof.

If $m_d=1$, then (4.2) immediately follows from (4.1) due to (4.3) and the fact that

$$M_d(b;\ m_1,\dots,m_{d-1},1) = M_{d-1}(b;\ m_1,\dots,m_{d-1},1).$$

In case $m_d = 2$ or 3 one can use the trivial estimate

$$M_d(b,m) \le \sum_{k_d=b_d+1}^{b_d+m_d} M_{d-1}\ (b_1,\dots,b_{d-1},k_d-1;\ m_1,\dots,m_{d-1},1)$$

and argue as in [7].

Now we assume, as the second induction hypothesis, that inequality (4.2) holds true for all values of the first 2d-1 arguments $b_1,\dots,b_d$; $m_1,\dots,m_{d-1}$ and for all values of the $(2d)^{th}$ argument less than m_d, $m_d \ge 4$.

The case $f(R) = f(b,m) = 0$ can be handled with ease since then $M(R) = 0$ a.e. Hence we assume that $f(R) \ne 0$. Then there exists an integer p_d, $1 \le p_d \le m_d$, such that

$$f(b;\ m_1,\dots,m_{d-1},p_d-1) \le \frac{1}{2} f(R) < f(b;\ m_1,\dots,m_{d-1},p_d), \tag{4.5}$$

the lefthand side being 0 in case $p_d = 1$. It is also convenient to set $S(b,m) = M(b,m) = 0$ if $m_j = 0$ for some j, $1 \le j \le d$.

Applying (2.2) for $j = d$ and taking (4.5) into account we obtain

$$f(b_1,\dots,b_{d-1},b_d+p_d;\ m_1,\dots,m_{d-1},m_d-p_d) \le$$

$$\le f(R) - f(b;\ m_1,\dots,m_{d-1},p_d) < \frac{1}{2} f(R).$$

The following three cases will be distinguished: $p_d = 1$, $2 \le p_d \le m_d-1$, and $p_d = m_d$.

<u>Case</u> (i): $2 \le p_d \le m_d-1$. Set

$$p'_d = \left[\frac{p_d-1}{2}\right] \quad \text{and} \quad q'_d = \begin{cases} p'_d & \text{if } p_d-1 \text{ is even,} \\ p'_d+1 & \text{if } p_d-1 \text{ is odd;} \end{cases}$$

$$p''_d = \left[\frac{m_d-p_d}{2}\right] \quad \text{and} \quad q''_d = \begin{cases} p''_d & \text{if } m_d-p_d \text{ is even,} \\ p''_d+1 & \text{if } m_d-p_d \text{ is odd.} \end{cases}$$

It is obvious that

$$p'_d + q'_d = p_d-1 \quad \text{and} \quad p''_d + q''_d = m_d - p_d.$$

Now, for $1 \le k_d \le m_d$, we can establish the following upper estimate:

$$M_{d-1}(b;\ m_1,\dots,m_{d-1},k_d) \le$$

$$\le \begin{cases} M_d(b;\ m_1,\dots,m_{d-1},p'_d) & \text{for } 1 \le k_d \le p'_d, \\ M_{d-1}(b;\ m_1,\dots,m_{d-1},q'_d)\ + & \\ \quad +\ M_d(b_1,\dots,b_{d-1},b_d+q'_d;\ m_1,\dots,m_{d-1},p'_d) & \text{for } q'_d \le k_d \le p_d-1, \\ M_{d-1}(b;\ m_1,\dots,m_{d-1},p_d)\ + & \\ \quad +\ M_d(b_1,\dots,b_{d-1},b_d+p_d;\ m_1,\dots,m_{d-1},p''_d) & \text{for } p_d \le k_d \le p_d+p''_d, \\ M_{d-1}(b;\ m_1,\dots,m_{d-1},p_d+q''_d)\ + & \\ \quad +\ M_d(b_1,\dots,b_{d-1},b_d+p_d+q''_d;\ m_1,\dots,m_{d-1},p''_d) & \text{for } p_d+q''_d \le k_d \le m_d \end{cases}$$

whence

$$M_d(b,m) \le M_{d-1}(b_1,\dots,b_{d-1},b_d;\ m_1,\dots,m_{d-1},q_d') + M_{d-1}(b_1,\dots,b_{d-1},b_d+q_d';$$

$$;m_1,\dots,m_{d-1},p_d-q_d') + M_{d-1}(b_1,\dots,b_{d-1},b_d+p_d;\ m_1,\dots,m_{d-1},q_d') +$$

$$+ \{M_d^{\gamma}(b_1,\dots,b_{d-1},b_d;\ m_1,\dots,m_{d-1},p_d') +$$

$$+ M_d^{\gamma}(b_1,\dots,b_{d-1},b_d+q_d';\ m_1,\dots,m_{d-1},p_d') +$$

$$+ M_d^{\gamma}(b_1,\dots,b_{d-1},b_d+p_d;\ m_1,\dots,m_{d-1},p_d'') +$$

$$+ M_d^{\gamma}(b_1,\dots,b_{d-1},b_d+p_d+q_d'';\ m_1,\dots,m_{d-1},p_d'')\}^{1/\gamma} = A_d + B_d, \tag{4.6}$$

where A_d denotes the sum of the first three terms and B denotes the fourth term on the righthand side of (4.6).

<u>Case</u> (ii): $p_d = 1$. Setting

$$p_d'' = \left[\frac{m_d-1}{2}\right] \quad \text{and} \quad q_d'' = \begin{cases} p_d'' & \text{if } m_d-1 \text{ is even,} \\ p_d''+1 & \text{if } m_d-1 \text{ is odd;} \end{cases}$$

we can estimate in a *simpler way*:

$$M_d(b,m) \le M_{d-1}(b_1,\dots,b_{d-1},b_d;\ m_1,\dots,m_{d-1},1) +$$

$$+ M_{d-1}(b_1,\dots,b_{d-1},b_d+1;\ m_1,\dots,m_{d-1},q_d'') +$$

$$+ \{M_d^{\gamma}(b_1,\dots,b_{d-1},b_d+1;\ m_1,\dots,m_{d-1},p_d'') +$$

$$+ M_d^{\gamma}(b_1,\dots,b_{d-1},b_d+q_d'';\ m_1,\dots,m_{d-1},p_d'')\}^{1/\gamma} = A_d' + B_d'. \tag{4.7}$$

<u>Case</u> (iii): $p_d = m_d$. Now

$$M_d(b,m) \le M_{d-1}(b_1,\dots,b_{d-1},b_d;\ m_1,\dots,m_{d-1},q_d'') +$$

$$+ M_{d-1}(b_1,\dots,b_{d-1},b_d+m_d-1;\ m_1,\dots,m_{d-1},1) +$$

$$+ \{M_d^{\gamma}(b_1,\dots,b_{d-1},b_d;\ m_1,\dots,m_{d-1},p_d'') +$$

$$+ M_d^{\gamma}(b_1,\dots,b_{d-1},b_d+q_d'';\ m_1,\dots,m_{d-1},p_d'')\}^{1/\gamma}, \tag{4.8}$$

where p_d'' and q_d'' are the same as in Case (ii).

The further reasoning closely follows that of the proof of Theorem 1 in [7]. We omit it.

<u>Proof of</u> (2.4). Without entering into details we note that we only have to modify estimates (4.6)-(4.8) in the following manner: *in Case* (i)

$$M_d(b,m) \le A_d'' + B_d,$$

where B_d is defined in (4.6) and

$$A_d'' = \{M_{d-1}^{\gamma}(b;\ m_1,\dots,m_{d-1},q_d') +$$

$$+ M_{d-1}^{\gamma}(b;\ m_1,\dots,m_{d-1},p_d) + M_{d-1}^{\gamma}(b;\ m_1,\dots,m_{d-1},p_d+q_d')\}^{1/\gamma};$$

in Case (ii)

$$M_d(b,m) \le A_d''' + B_d',$$

where B_d' is defined in (4.7) and

$$A_d''' = \{M_{d-1}^{\gamma}(b;\ m_1,\dots,m_{d-1},1) + M_{d-1}^{\gamma}(b;\ m_1,\dots,m_{d-1},q_d'')\}^{1/\gamma};$$

and a similar modification of (4.8) in Case (iii).

Thus, by a double induction, one can prove both (2.3) and (2.4) for each $m_d = 1,2,\dots$ and for each $d = 1,2,\dots$.

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20. ABSTRACT

In a recent paper we presented a general method of how to obtain an upper estimate *for a fixed moment of the maximum of partial sums of a single series* in terms of the given "a priori" upper estimate for the same moment of the partial sums. Now we extend this method from single series to multiple series. A number of special cases interesting in themselves are included.

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